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Diophantine Collinearity and Lepton Masses

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The combined requirements of Diophantine quantization (integral or possibly half-integral solutions of mass equations) and collinearity (with respect to a rational quantum number which need not be specified explicitly) lead to limitations on heavy lepton masses, with four allowed values in the 13.3–16.9-GeV range.

Diophantine quantization (Pease, 1970) postulates that equations involving particle masses must be solved in integers (or possibly half-integers), in analogy to the integer solutions (3,4,5), (5,12,13), etc., of the Pythagorean equation $x^2+y^2=z^2$. This principle yielded surprising results when applied to the Gell-Mann–Okubo equation (Gell-Mann, 1961;[1] Okubo, 1962) for meson masses

$$\pi^2+3\eta^2=4K^2 \tag{1}$$

for here the lowest nontrivial integer solution was (2,8,7), in close proportion to the respective particle masses (Bricman et al., 1978) of 135–140, 549, and 494–498 MeV. This result also lent strength to the proposal (Nambu, 1952) of a unit mass of 70 MeV($=m_e c^2/\alpha$), and indicated perhaps that the Gell-Mann–Okubo equation was more fundamental than had been realized.

It is also interesting that the massive leptons so far discovered, μ and τ, have respective masses (Bricman et al., 1978; Flügge, 1979) of 106 Mev and approximately 1782 MeV, quite close to 3/2 and 51/2 multiples of 70 MeV.

The aim of this paper is to set up reasonable necessary conditions for the existence of leptons of higher mass and to predict the resulting allowed

[1]The use of squared masses was suggested by R. P. Feynman.

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spectrum. To do this it is useful to postulate that a set of particles possessing many properties in common obey Diophantine collinearity, i.e., that (a) in a generalization of various extant mass relations, both from group theory and from strong-interaction dynamics, the sth powers of their mass numbers lie along the straight line

$$m_i^s = a + bZ_i \tag{2}$$

where the Z_i are quantum numbers, as yet unspecified, rational but with not too large a denominator, and that (b) only those few integral (or half-integral) values of the m_i which happen to correspond to suitable values of the Z_i can be associated with the set.

Three such mass numbers give rise to the collinearity condition

$$\frac{m_k^s - m_i^s}{m_j^s - m_i^s} = \frac{Z_k - Z_i}{Z_j - Z_i} \tag{3}$$

It is possible, if desired, to use standard number-theoretic methods to derive, for various values of the Z ratio on the right-hand side of (3), sets of solutions (m_i, m_j, m_k). But here we shall be given an m_i and an m_j and asked to find suitable m_k's; to do this systematically, it is more useful to develop a few simple results of collinearity theory.

Define a *family* $F(m_1, D, s)$ as the ordered set of those nonnegative integers $0 \leqslant m_1 < m_2 < \cdots$ which, for given integers m_1, D, and s, obey the relation

$$m_k^s - m_1^s \equiv 0 (\mathrm{mod}\, D) \tag{4}$$

We note that

$$F(m_1, D, s) \subseteq F(m_1, D' | D, s) \tag{5}$$

where $D'|D$ means "D' is a divisor of D."

With the help of a little elementary number theory (e.g., Hardy and Wright, 1960), we can derive three useful theorems about family membership, bearing in mind that family membership does not guarantee physical existence:

Theorem I. If p is any integer such that $pD+m \geqslant 0$, and $m \in F$, then $(pD+m) \in F$. This has two consequences: (a) sequences of family members repeat themselves with periodicity D, and (b) each portion of the mass spectrum of width D has the same number of members, so the average permitted mass density is a constant.

Theorem II. For s even, if $m \in F$, then $(D-m) \in F$.

Theorem III. For $s=2$, p and q positive integers, if $m \in F, q|2m$ and$q^2|D$,, then $[(pD/q)+m] \in F$.

By doing a little systematic searching and using these theorems, it is not difficult to locate all members of a family, given m_1, D, and s.

We note that both the μ and τ masses are very close to integral multiples of $(3/2)\times 70$ MeV. Hence to apply our collinearity theorems, which for simplicity were derived on the basis of integral mass numbers, we set $\mu = m_1 \times (3/2) \times 70$ MeV, $\tau = m_2 \times (3/2) \times 70$ MeV, which gives $m_1 = 1$, $m_2 = 17$; all resulting heavy lepton masses will be integral multiples of $(3/2)\ \times 70$ MeV.

If we choose $s=2$ for various theoretical and empirical reasons, we see from (4) that the largest possible value of D is 288. With this value, a brief systematic search and application of the collinearity theorems give the following as the next four allowed heavy lepton masses:

$$127 \times (3/2) \times 70 \text{ MeV} = 13{,}335 \text{ MeV} \tag{6a}$$

$$143 \times (3/2) \times 70 \text{ MeV} = 15{,}015 \text{ MeV} \tag{6b}$$

$$145 \times (3/2) \times 70 \text{ MeV} = 15{,}225 \text{ MeV} \tag{6c}$$

$$161 \times (3/2) \times 70 \text{ MeV} = 16{,}905 \text{ MeV} \tag{6d}$$

The complete mass spectrum consists of the four sets of multiples $(1+144n)$, $(17+144n)$, $(127+144n)$, and $(143+144n)$ of $(3/2)\times 70$ MeV. It must be emphasized that these are *allowed* values and need not, on the basis of the theory so far developed, exist physically.

It is of course quite possible to use a smaller D as long as it is a divisor of 288. From (5), we see that the masses in (6) are still permitted, but others may be also. For the next lower value of D, 144, lower masses of 5775, 7455, 7665, and 9345 MeV are also permitted, though if the lowest three, at least, were physical masses, it is quite possible that they might have been discovered already.

The masses in (6) would correspond to quite high quantum numbers, with Z ratios, from (3), of 56, 71, 73, and 90, respectively, but such values could well appear from Casimir invariants of high-rank Lie algebras (e.g., Umezawa, 1963, 1964).

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